Revista Facultad de Ciencias Básicas
2021
Vol. 17(1)
enero - junio ■ ISSN: 1900-4699 · e-ISSN: 2500-5316 ■ pp. 101-114

Editorial Neogranadina

DOI: https://doi.org/10.18359/rfcb.5102

Some Classical Methods in the Analysis of an *Aedes aegypti* Model*

Julián Alejandro Olarte García[a] ■ Aníbal Muñoz Loaiza[b]

Abstract: the Taylor series approximation is often used to convert non-linear dynamical systems to linear systems, while the Hartman-Großman theorem analyzes the local qualitative behavior of the non-linear system around a hyperbolic equilibrium point. The global stability of an equilibrium point in the Lyapunov sense is based on the principle that if the equilibrium point is disturbed and the flow of the system is dissipative, then the system must be stable. This article applies these methods to an ecological *Aedes aegypti* model, whose local and global stability are characterized by a population growth threshold. In conclusion, the classical theory of dynamical systems, validated computationally, yields theoretical results in favor of controlling the local population of *Aedes aegypti*. It becomes usable if the proposed model is reinforced with the estimation of the parameters that describe the relationships between stages (aquatic and aerial) of the mosquito population and the inclusion of vector control strategies to protect people from the viruses transmitted by *Aedes aegypti*.

Keywords: *Aedes aegypti*; Taylor series; Hartman-Großman theorem; Lyapunov function

Received: 7th August 2020

Accepted: 9th April 2020

Available online: 19th November 2021

How to cite: J. A. Olarte García y A. Muñoz Loaiza, «Algunos métodos clásicos para el análisis de un modelo de *Aedes aegypti*», Rev. Fac. Cienc. Básicas, vol. 17, n.º 1, pp. 101-114, nov. 2021.

* Research article

a MS in Biomathematics, Universidad del Quindío. BS in Mathematics, Universidad del Quindío. Member of the research group "Grupo de Modelación Matemática en Epidemología", Universidad del Quindío, Armenia (Quindío), Colombia.
E-mail: jaolarteg@uniquindio.edu.co ORCID: https://orcid.org/0000-0002-8972-582X

b PhD in Mathematical Sciences, FCFM-BUAP, Mexico. Specialist in Biomathematics, Universidad del Quindío. BS in Mathematics, Universidad del Quindío. Member of the research group "Grupo de Modelación Matemática en Epidemología", Universidad del Quindío, Armenia (Quindío), Colombia.
E-mail: anibalml@hotmail.com ORCID: https://orcid.org/0000-0002-9294-1363

Algunos métodos clásicos para el análisis de un modelo de Aedes aegypti

Resumen: la aproximación de la serie de Taylor se utiliza con frecuencia para convertir sistemas dinámicos no lineales en sistemas lineales, mientras que el teorema de Hartman-Großman analiza el comportamiento cualitativo local del sistema no lineal en relación con un punto de equilibrio hiperbólico. La estabilidad global de un punto de equilibrio en el sentido de Lyapunov tiene como base el principio de que si el punto de equilibrio se altera y el flujo del sistema es disipativo, entonces el sistema debe ser estable. En este artículo, aplica estos métodos a un modelo ecológico de *Aedes aegypti,* cuya estabilidad local y global se caracteriza por un umbral de crecimiento poblacional. Se concluye que la teoría clásica de los sistemas dinámicos, se valida computacionalmente, arroja resultados teóricos a favor del control de la población local de *Aedes aegypti,* que se hace utilizable en la práctica si se refuerza el modelo propuesto con la estimación de los parámetros que describen las relaciones entre las etapas (acuáticas y aéreas) que conforman la población de mosquitos y la inclusión de estrategias de control de vectores para proteger a las personas de los virus transmitidos por el *Aedes aegypti.*

Palabras clave: *Aedes aegypti*; serie de Taylor; teorema de Hartman-Großman; función de Lyapunov

Alguns métodos clássicos para a análise de um modelo de Aedes aegypti

Resumo: a aproximação da série de Taylor é frequentemente usada para converter sistemas dinâmicos não lineares em sistemas lineares, enquanto o teorema de Hartman-Großman analisa o comportamento qualitativo local do sistema não linear em relação a um ponto de equilíbrio hiperbólico. A estabilidade geral de um ponto de equilíbrio no sentido Lyapunov baseia-se no princípio de que: se o ponto de equilíbrio for alterado e o fluxo do sistema for dissipativo, então o sistema deve ser estável. Neste artigo, são aplicados esses métodos a um modelo ecológico de *Aedes aegypti,* cuja estabilidade local e global é caracterizada por um limiar de crescimento populacional. Conclui-se que a teoria clássica dos sistemas dinâmicos, validada computacionalmente, produz resultados teóricos em favor do controle da população local de *Aedes aegypti,* que se torna utilizável na prática se o modelo proposto for reforçado com a estimativa dos parâmetros que descrevem as relações entre as etapas (aquáticas e aéreas) que compõem a população de mosquitos e a inclusão de estratégias de controle vetorial para proteger pessoas dos vírus transmitidos pelo *Aedes aegypti.*

Palavras-chave: *Aedes aegypti*; série de Taylor; teorema de Hartman-Großman; funções de Lyapunov

Introduction

Several phenomena that occur in nature are modeled by non-linear systems of ordinary differential equations, covering a wide field of applications. In epidemiology, for example, non-linear models serve to study the behavior of plants, animals, and humans; in plants, dynamics include interactions between crops and pests [1]-[3]. In animals, the behavior of epizootics such as foot-and-mouth disease, ovine brucellosis, and avian influenza are modeled [4]-[6]. For humans, the dynamics of infectious diseases such as malaria, influenza, dengue, Zika, HIV, among others, are described [7]-[10]. Economically, Lotka-Volterra non-linear models explain the cyclical behavior of economic growth [11]. However, most of these events must be approached from a qualitative approach since it is difficult to find analytical solutions.

In 1715, the mathematician Brook Taylor formally introduced the concept of Taylor series as the development of a sufficiently differentiable function in an infinite series within a domain, whose coefficients involve the successive derivatives of the function [12]. It is used in several branches of science. For example, it is employed to formulate, extend, or interpret typical models of heat conduction in physics, to develop finite difference schemes or numerical integration methods in engineering, and to implement computational methods that allow for numerical solutions to ordinary and partial differential equations in dynamic systems [13]-[16].

Taylor series transforms a non-linear system of differential equations into a linear system (Jacobian method); linearization is local and approximated, that is, valid only for a region around a given equilibrium point [17]. A significant result that justifies concluding about a non-linear system from a linearized system is known as the Hartman-Grobman theorem. This theorem says that the solutions to an autonomous system of ordinary differential equations in a steady-state neighborhood look qualitatively just like the solutions to the linearized system near the origin only when the equilibrium is hyperbolic, that is, when none of the Jacobian matrix eigenvalues have zero real parts [18].

Another powerful method for the qualitative analysis of a dynamical system was developed in 1892 by the Russian mathematician and engineer A. M. Lyapunov in his doctoral thesis on the stability of motion [19]. However, the notion of an auxiliary function has found a more comprehensive range of application, and Lyapunov functions may be used to achieve many diverse objectives. For instance, this method may be applied to estimate the rate of convergence to a steady state or the size of a basin of attraction. It has also been employed to prove theorems (*e.g.*, Hopf bifurcation theorem) [19]-[20].

The structure of the present work is as follows: the next section provides some concepts relevant to this study. The third section deals with the approach of the *Aedes aegypti* model. The analysis of qualitative properties and numerical results of the model are presented in the fourth and fifth sections. The paper ends with a brief discussion.

Preliminaries

The following definitions and theorems of the dynamical systems theory are required to study ecological dynamics formally.

Definition 1. A *homeomorphism* between two topological spaces, X and Y, is a one-one function, H from Y onto X, such that H and the inverse of H are continuous [21].

Definition 2. The non-linear system of autonomous differential equations, $\dot{x} = f(x)$, is said to be *topologically equivalent* to the linear system $\dot{x} = Ax$ in the neighborhood of the origin if a homeomorphism H applies to an open U onto an open V containing the origin. The latter sends trajectories of $\dot{x} = f(x)$ in U onto trajectories of $\dot{x} = Ax$ in V and preserves their orientation over time [17].

Theorem 1. [*Taylor formula*] Let $f(x)$ be a function of C^{n+1} class in an open $U \subseteq R^n$ and let $(h, x_0) \in U \times U$. Then, for each $x \in U$ with $x = x_0 + h$

$$f(x) = f(x_0) + \frac{1}{1!}\sum_{i_1=1}^{n} \frac{\partial f(x_0)}{\partial x_{i_1}} \cdot h_{i_1} + \frac{1}{2!}\sum_{i_1,\, i_2=1}^{n} \frac{\partial^2 f(x_0)}{\partial x_{i_2}\partial x_{i_1}} \cdot (h_{i_1}h_{i_2})$$
$$+\frac{1}{3!}\sum_{i_1,\, i_2, i_3=1}^{n} \frac{\partial^3 f(x_0)}{\partial x_{i_3}\partial x_{i_2}\partial x_{i_1}} \cdot (h_{i_1}h_{i_2}h_{i_3}) + \cdots$$
$$+\frac{1}{(n-1)!}\sum_{i_1, i_2, \dots, i_{n-1}=1}^{n} \frac{\partial^{n-1} f(x_0)}{\partial x_{i_{n-1}}\partial x_{i_{n-2}} \dots \partial x_{i_1}} \cdot (h_{i_1}h_{i_2} \dots h_{i_{n-1}})$$
$$+\frac{1}{n!}\sum_{i_1, i_2, \dots, i_n=1}^{n} \frac{\partial^{n} f(x_0)}{\partial x_{i_n}\partial x_{i_{n-1}} \dots \partial x_{i_1}} \cdot (h_{i_1}h_{i_2} \dots h_{i_n}) + R_{n+1}(h)$$

where R_{n+1} (h) is the remainder (or error) of the Taylor polynomial of $f(x)$ at $x = x_0$, an infinitesimal of order greater or equal to $n + 1$ at x_0, in the sense that $\lim_{h \to x_0} \frac{R_{n+1}(h)}{\|h\|^{n+1}} = 0$. This remainder admits the Lagrange form, which is given by

$$R_{n+1}(h) = \frac{1}{(n+1)!}\sum_{i_1, i_2, \dots, i_{n+1}=1}^{n} \frac{\partial^{n+1} f(\xi)}{\partial x_{i_{n+1}}\partial x_{i_n} \dots \partial x_{i_1}} \cdot (h_{i_1}h_{i_2} \dots h_{i_{n+1}})$$
$$\frac{1}{(n+1)!}\sum_{i_2, \dots, i_{n+1}=1}^{n} \frac{\partial^{n} f(\xi)}{\partial x_{i_{n+1}}\partial x_{i_n} \dots \partial x_{i_2}} \cdot (h_{i_2}h_{i_3} \dots h_{i_{n+1}}) \left(\frac{\partial f(\xi)}{\partial x_1} \cdot h_1 + \frac{\partial f(\xi)}{\partial x_2} \cdot h_2 + \cdots \right.$$
$$\left. + \frac{\partial f(\xi)}{\partial x_{\mathrm{n}-1}} \cdot h_{\mathrm{n}-1} + \frac{\partial f(\xi)}{\partial x_{\mathrm{n}}} \cdot h_{\mathrm{n}}\right)$$

with $\xi = x_0 + \xi h$, $0 < \xi < 1$ [22].

Theorem 2. [*Hartman-Grobman theorem*] Let E, U, and V be open subsets of R^n each containing the origin. Let $I_0 \subset R$ be an open interval containing zero. Let $f \in C^1$ (E) and ϕ_t (x) be the flow of the non-linear system $\dot{x} = f(x)$. Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ (Jacobian matrix) has no eigenvalue with zero real parts. Then, there is a homeomorphism $H : U \longmapsto V$ such that for each $x_0 \in U$, and there is I_0 such that for all $x_0 \in U$ and $t_0 \in I_0$, $H \circ \phi t\ (x_0) = e^{At} H(x_0)$, *i.e.*, H maps trajectories of $\dot{x} = f(x)$near the origin onto trajectories of $\dot{x} = Ax$ near the origin and preserves the parameterization over time [17].

Lemma 1. There is a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that $H \circ T = L \circ H$, where L is the linear flow and T the non-linear flow at $t = 1$, *i.e.*, $L(x) = e^A x$ and locally $T(x) = \phi_1(x)$ [17].

Theorem 3. Let $x = x^*$ be an equilibrium point of the vector field $\dot{x} = f(x)$ and let $L_1 : U \longmapsto R^n$ be a C^1 function defined in a neighborhood U of x^* such that i) $L_1(x^*) = 0$ and $L_1(x) > 0$. If $x \neq x^*$ and ii) $L_1(x) \leq 0$ in $U - \{x^*\}$ then, x^* is stable. Moreover, if iii) $\dot{L}_1(x^*) < 0$ in $U - \{x^*\}$, then, x^* is asymptotically stable [19], [23].

Theorem 4. Let $\Omega \subset U$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let L_1 be the same function defined in Theorem 3. Let Θ be the set of all points in Ω where $\dot{L}_1 = 0$. Let S be the largest invariant set in Θ. Then, every solution starting in Ω approaches S as $t \longrightarrow \infty$ [23]-[24].

Dynamic system

This study considers the autonomous version of the ecological *Aedes aegypti* model proposed in [25], which is originally semi-coupled with the differential equations of the breeding sites through the variable carrying capacity of the aquatic *Aedes aegypti* population. This model has the following variables at time t and parameters: $x_1 \equiv x_1(t)$, the average number of female mosquitoes in the aerial phase; $x_2 \equiv x_2(t)$, the average number of female mosquitoes in the aquatic phase; ω, the rate of transition from immature stage to adult stage; ϵ, adult mosquito mortality rate; ϕ, per head oviposition rate; f, the fraction of eggs producing female mosquitoes; π, the mortality rate of immature stages; K, the carrying capacity of aquatic mosquitoes in epidemiologically relevant breeding sites. The non-linear dynamic system that models population growth is:

$$\frac{dx_1(t)}{dt} = \omega x_2(t) - \epsilon x_1(t) \equiv g_1(x_1, x_2)$$

$$\frac{dx_2(t)}{dt} = f\phi x(t)(1 - \frac{x_2(t)}{\underline{K}}) - (\pi + \omega)x_2(t) \ \equiv \ g_2(x_1, x_2) \quad (1)$$

$$x_1(0) = x_{10} \ \geq \ 0, x_2(0) = x_{20} \ \geq \ 0$$

where $\omega > 0$, $\epsilon > 0$, $\phi > 0$, $\pi > 0$, $\underline{K} > 0$, and $0 < f < 1$. This model makes biological sense in the positively invariant region:

$$\Xi = \left\{ [x_1 \quad x_2]^\top \in R^2 : 0 \ \leq \ x_1 \leq \frac{\omega}{\epsilon}\underline{K}, 0 \ \leq \ x_2 \leq \underline{K} \right\} \quad (2)$$

A system of non-linear differential equations with constant coefficients has time-independent solutions, that is, constant solutions over time that play an important role in the long-term behavior of other non-stationary solutions. Linearizing the system is necessary to know the equilibrium solutions of system (1), which are solutions of the following homogeneous algebraic system:

$$0 = \omega x_2 - \epsilon x_1,$$
$$0 = f\phi x_1 \left(1 - \frac{x_2}{\underline{K}}\right) - (\pi + \omega)x_2$$

An equilibrium solution is without colonization of mosquitoes:

$$x^{(0)} = [0 \quad 0]^\top, \quad (3)$$

and the other with colonization:

$$x^{(1)} = \left[\underline{x_1} \quad \underline{x_2}\right]^\top = \left[(\omega/\epsilon)\underline{x_2} \quad (1 - 1/R_m)\underline{K}\right]^\top \quad (4)$$

The equilibrium $x^{(1)}$ makes biological sense if

$$x_2 \ \geq \ 0 \ \Leftrightarrow \ 1 - \frac{\epsilon(\pi + \omega)}{f\phi\omega} \ \geq \ 0 \ \Leftrightarrow \ R_m = \frac{f\phi\omega}{\epsilon(\pi + \omega)} \ \geq \ 1 \quad (5)$$

where R_m is a population growth threshold of *Aedes aegypti*. It is proved in [25] that the absence of the mosquito population is stable if $R_m \leq 1$; otherwise ($R_m > 1$), the presence of the mosquito population in a region is unstable.

For numerical simulation purposes, the values of the entomological parameters were determined by evaluating at 22 °C the polynomial functions derived from empirical data in [26], while f and K are hypothetical (Table 1).

Table 1. Parameter values

Parameter	ϕ	ω	ϵ
Average value	3.985816	0.1017712	0.36082704
Parameter	π	$\underline{K}$	(f, R_m)
Average value	0.42800864	500000	(0.5, 0.3888), (0.005, 38.88)

Note: Parameter unit *day* $^{(-1)}$ except for $\underline{K}$, f, and R_m that are dimensionless.

Source: Own elaboration

Local qualitative analysis

Taylor series approximation

The idea of linearization is to shift the equilibrium to zero. Thus, the perturbation (sufficiently small quantity) that gives the deviation of a solution to (1) from a generic equilibrium $x^\star$ is denoted by $u(t) = x(t) - x^\star$ where $u \equiv u(t) = [u_1(t)\ u_2(t)]^\top$, $x \equiv x(t) = [x_1(t) \ \ x_2(t)]^\top$, and $x^\star = [x_1^\star(t) \ \ x_1^\star(t)]^\top$.

Denote the right side of (1) as $g \equiv g(x) = [g_1(x_1, x_2)\ g_2(x_1, x_2)]^\top$, replace $x(t) = u(t) + x^*$ in the original system, and expand g_m for each $m = 1, 2$ around x^* in a Taylor series under Theorem 1. Note that g_m has at least continuous third-order partial derivatives:

$$g_m(u + x^*) = g_m(x^*) + \frac{\partial g_m(x^*)}{\partial\{x\}_1}u_1 + \frac{\partial g_m(x^*)}{\partial x_2}u_2 + \frac{\partial^2 g_m(x^*)}{\partial x_1^2}\frac{u_1^2}{2} + \frac{\partial^2 g_m(x^*)}{\partial x_2 \partial x_1}u_1u_2 + \frac{\partial^2 g_m(x^*)}{\partial x_2^2}\frac{u_2^2}{2} + \cdots \tag{6}$$

where $g(x^*) = 0$ for $m = 1, 2$. If the vector field is approximated to first order, it will yield the following linear system at u_1 and u_2:

$$\begin{gathered} \frac{du_1}{dt} = -\epsilon u_1 + \omega u_2 \equiv h_1(u_1, u_2) \\ \frac{du_2}{dt} = f\phi\left(1 - \frac{x_2^*}{\underline{K}}\right)u_1 - \left(f\phi\frac{x_1^*}{\underline{K}} + \pi + \omega\right)u_2 \equiv h_2(u_1, u_2) \\ u_1(0) = x_{10} - x_1^*, \qquad u_2(0) = x_{20} - x_2^* \end{gathered} \tag{7}$$

where $\omega, \epsilon, \phi, \pi, \underline{K} > 0$; $0 < f < 1$; $x_{10} \geq 0$, $x_{20} \geq 0$. Its matrix form is $\frac{du}{dt} = Ju$, where:

$$J \equiv J(x^*) = \left[\frac{\partial g_i}{\partial x_j} \mid 1 \leq i \leq 2, 1 \leq j \leq 2\right]$$

is the Jacobian matrix.

Similarly, if the vector field is approximated to second order, it will yield the following non-linear system at u_1 and u_2:

$$\begin{gathered} \frac{du_1}{dt} = -\epsilon u_1 + \omega u_2 \equiv f_1(u_1, u_2) \\ \frac{du_2}{dt} = f\phi\left(1 - \frac{x_2^*}{\underline{K}}\right)u_1 - \left(f\phi\frac{x_1^*}{\underline{K}} + \pi + \omega\right)u_2 - \frac{f\phi}{\underline{K}}u_1u_2 \equiv f_2(u_1, u_2) \\ u_1(0) = x_{10} - x_1^*, \qquad u_2(0) = x_{20} - x_2^*. \end{gathered} \tag{8}$$

Written in the matrix form, it is $\frac{du}{dt} = Ju + \frac{1}{2}u^\top Hu$, where:

$$H \equiv H(x^*) = \left[\frac{\partial^2 g}{\partial x_i x_j} \mid 1 \leq i \leq 2, 1 \leq j \leq 2\right]$$

is the Hessian matrix. Note that every g_m has finite expansion in a Taylor series because they are polynomial functions.

As with system (1), the trajectories of systems (7) and (8) exist and do not leave the compact

$$\Pi = \left\{ [u_1\ \ u_2]^\top \in R^2 : -x_1^* \leq u_1 \leq \frac{\omega}{\epsilon}\underline{K} - x_1^*, -x_2^* \leq u_2 \leq \underline{K} - x_2^* \right\} \tag{9}$$

The surfaces of Fig. 1 have been generated for $R_m > 1$ and show the approximations to $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ geometrically when (u_1, u_2) is

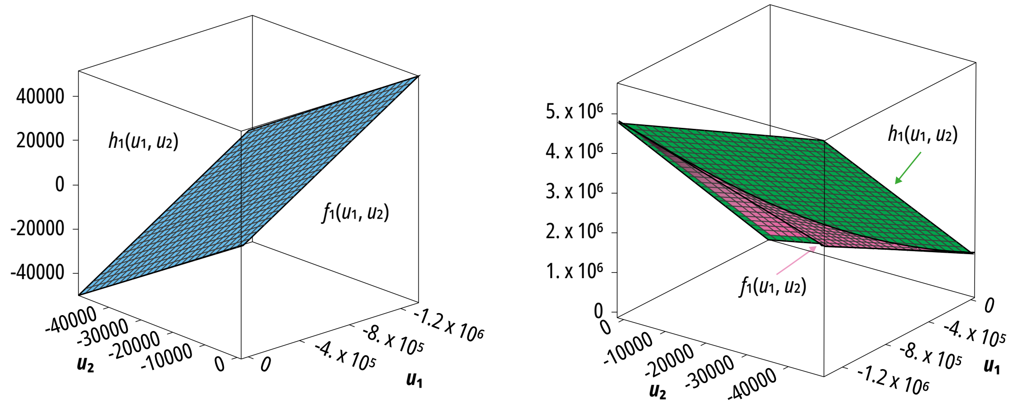

Fig. 1. Graphs of Taylor polynomials that approximate the scalar fields of system (1).

Source: Own elaboration

very close to (0, 0). Naturally, the bilinearity of $f_1(u_1, u_2)$ implies that its Taylor polynomial is the same function $f_1(u_1, u_2)$ in the domain Π, whereas for $f_2(u_1, u_2)$ the nonlinearity $u_1 u_2$ causes the approximation to become acceptable in a δ-neighborhood of the origin, whose error will not exceed $f\phi\delta^2/\underline{K}$. Note that disturbances can increase or decrease over time; for example, $h_1(u_1, u_2) \geq 0$ and $h_2(u_1, u_2) \leq 0$, which limit the prediction of whether the mosquito population will tend to reach equilibrium with colonization still at population densities far from this equilibrium.

Construction of a homeomorphism

Given the complexity of deducing the analytical solution of initial value problem (8) required to apply Theorem 2, the steps for constructing homeomorphism H in Theorem 2 are indicated in the case of non-linear system (8). Firstly, because $f = [f_1\ f_2]^\top \in C^1(\Pi)$ with Π occurred in (9) and $f(0) = 0$, this system can be written as

$$\dot{u} = Au + F(u),$$

where:

$$J = \left[\frac{\partial f_i(0)}{\partial u_j}\right], F(u) = [F_1(u)$$

$$F_2(u)]^\top = f(u) - Ju, F(0) = 0, \text{ and } \left[\frac{\partial F_i(0)}{\partial u_j}\right] = 0.$$

Furthermore, there is an $n \times n$ (2×2) invertible matrix E whose columns are the eigenvectors of J such that

$$M = E^{-1}AE = [P \quad 0\ ;\ 0 \quad Q]$$

where the eigenvalues of the $k \times k$ (1×1) matrix P have negative real parts and the eigenvalues of $(n - 1) \times (n - k)$ (1×1) matrix Q have positive real parts. If $v = E^{-1}u$, system (8) then will have the form

$$\dot{v} = Mv + G(v) \tag{10}$$

where $G(v) = E^{-1} F(Ev)$. Explicitly:

$$J(x^*) = \left[-\epsilon \quad \omega\ ;\ f\phi\left(1 - \frac{x_2^*}{\underline{K}}\right) \quad -\left(f\phi\frac{x_1^*}{\underline{K}} + \pi + \omega\right)\right]$$

Its trace and determinant are:

$$tr\, J(x^*) = -\epsilon - \left(f\phi\frac{x_1^*}{\underline{K}} + \pi + \omega\right),$$

$$det\, J(x^*) = \epsilon(\pi + \omega)\left(2R_m\frac{x_2^*}{\underline{K}} + 1 - R_m\right)$$

And its eigenvalues are:

$$\lambda_m = \frac{tr\, J(x^*) \pm \sqrt{trJ(x^*)^2 - 4\, det\, J(x^*)}}{2}$$

$$(m = 1, 2)$$

Manipulating the discriminant of λ_m algebraically, it yields

$$tr^2 J(x^*) - 4 det\, J(x^*) =$$

$$\left(f\phi\frac{x_1^*}{K} + \pi + \omega - \epsilon\right)^2 + f\phi\omega\left(1 - \frac{x_2^*}{K}\right) \geq 0, \tag{11}$$

i.e., $\lambda_m \in R$. Additionally, inequality (11) is strict $(\lambda_1 \neq \lambda_2)$ because if $[x_1^*(t)\ x_2^*(t)]^\top$ (that is, (3) and (4)) is replaced in (11), then $f\phi\omega(1 - x_2^*/\underline{K}) \neq 0$.

Concerning the sign of the real part of the eigenvalues, $sgn(Re(\lambda_m)) = sgn(\lambda_m)$, Table 2 contains possibilities P1 to P6, of which P4 is not biologically admitted.

Table 2. Conditions that determine the sign of the real eigenvalues of J

$R_m = \alpha$	$\alpha < 1$	$\alpha = 1$	$\alpha > 1$
$x = x^*$			
$x^* = x^{(0)}$	$P1$: $\lambda_1 < 0, \lambda_2 < 0$	$P2$: $\lambda_1 = 0, \lambda_2 < 0$	$P3$: $\lambda_1 > 0, \lambda_2 < 0$
$x^* = x^{(1)}$	$P4$: $\lambda_1 < 0, \lambda_2 < 0$	$P5$: $\lambda_1 = 0, \lambda_2 < 0$	$P6$: $\lambda_1 > 0, \lambda_2 < 0$

Source: Own elaboration

A pair of corresponding eigenvectors to λ_1 and λ_2 is given by

$$p_1 = \begin{bmatrix} \frac{\omega}{\lambda_1+\epsilon} & 1 \end{bmatrix}^\top \text{ and } p_2 = \begin{bmatrix} \frac{\omega}{\lambda_2+\epsilon} & 1 \end{bmatrix}^\top \tag{12}$$

Thus, J is diagonalizable. Its diagonalization along with the coordinate transformation converts system (8) into (10) or equivalently:

$$\begin{gathered} \dot{v}_1 = \lambda_1 v_1 - \frac{f\phi\omega\left((\lambda_2+\epsilon)v_1{}^2 + (\lambda_1+\lambda_2+2\epsilon)v_1v_2 + (\lambda_1+\epsilon)v_2{}^2\right)}{(\lambda_1-\lambda_2)(\lambda_2+\epsilon)\underline{K}} \\ \dot{v}_2 = \lambda_2 v_2 + \frac{f\phi\omega\left((\lambda_2+\epsilon)v_1{}^2 + (\lambda_1+\lambda_2+2\epsilon)v_1v_2 + (\lambda_1+\epsilon)v_2{}^2\right)}{(\lambda_1-\lambda_2)(\lambda_1+\epsilon)\underline{K}} \\ [v_1(0) \quad v_2(0)]^\top = [v_{10} \quad v_{20}]^\top = E^{-1}[u_{10} \quad u_{20}]^\top. \end{gathered} \tag{13}$$

where $M = [p_1 \;\; p_2]$ and each p_m is in (12). Theorem 2 shows that near the hyperbolic equilibrium point $v_1 = v_2 = 0$, non-linear system (13) has the same qualitative structure as the linear system.

$$\begin{aligned} &\dot{v}_1 = \lambda_1 v_1, \;\; \dot{v}_2 = \lambda_2 v_2 \\ &[v_{10} \quad v_{20}]^\top = E^{-1}[u_{10} \quad u_{20}]^\top. \end{aligned} \tag{14}$$

Fig. 2 allows comparing the behavior of the non-linear system with the associated linear system in a short time interval if P1, P3, and P6 specified in Table 2 are fulfilled. The trajectories of the states in both dynamical systems almost adjust to each other with very little variability at each moment. Moreover, R_m is also a key parameter in the linear model: if P1 and P6 are met, then the trivial equilibrium point will be globally stable (a, b, e, and f). Otherwise, it will be unstable (c and d).

The solid curves in Fig. 3 can be obtained by applying the transformation of coordinates $u = Ev$. There is again a continuous one-to-one map of a δ-neighborhood of 0 (N_δ) onto an open set containing 0, H: $N_\delta \longmapsto U$, which maps 'dotted' trajectories in N_δ onto 'solid' trajectories in U and preserves the direction of the flow along the trajectories (Theorem 2).

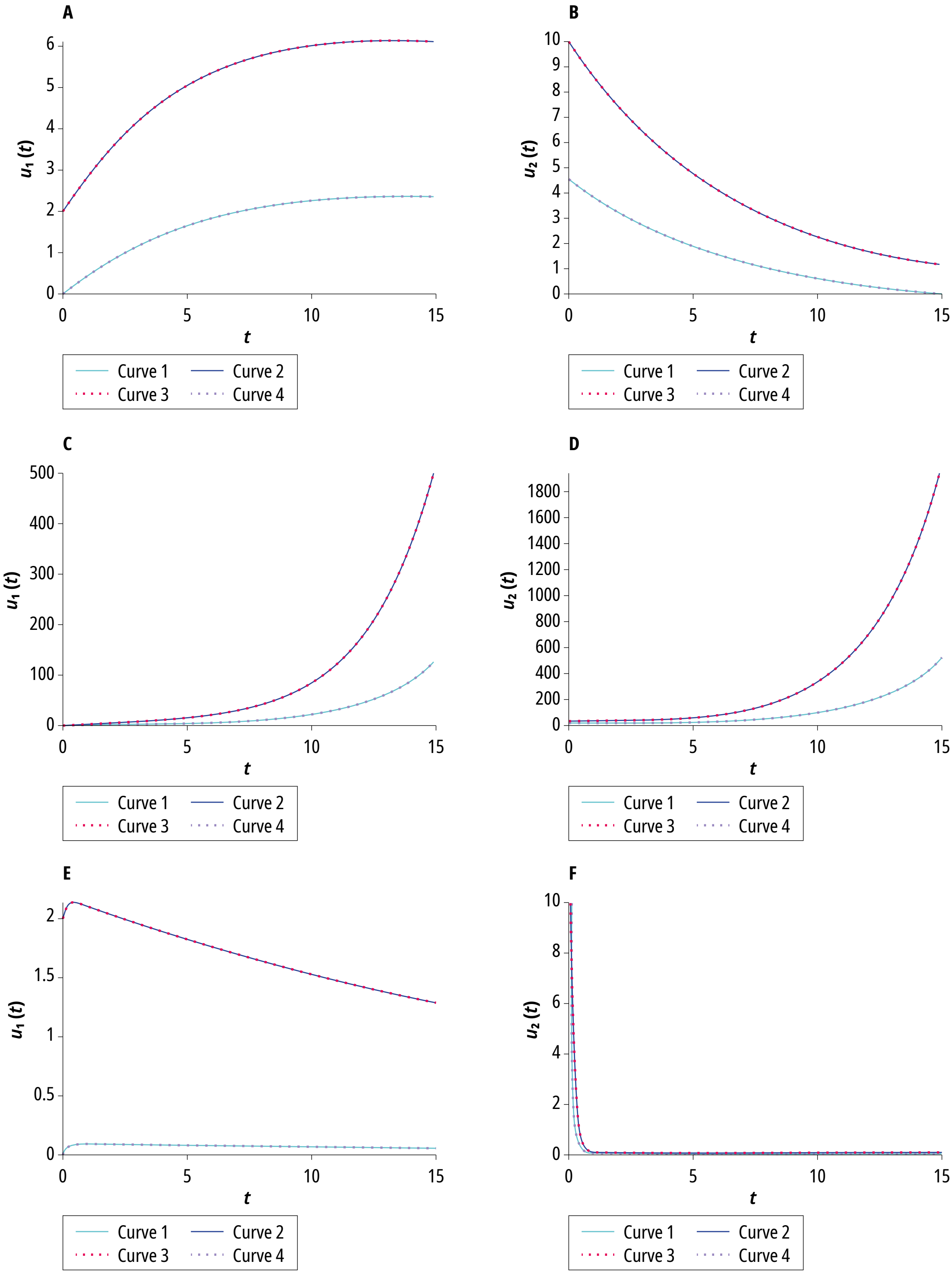

Fig. 2. Solution curves of non-linear system (8) (solid line) and linear approximation (7) (dotted line) with the initial conditions $u_0 = [0\ \ 5]^T$ and $[2\ \ 10]^T$, when P1 (a and b), P3 (c and d), or P6 (e and f) are met.

Source: Own elaboration

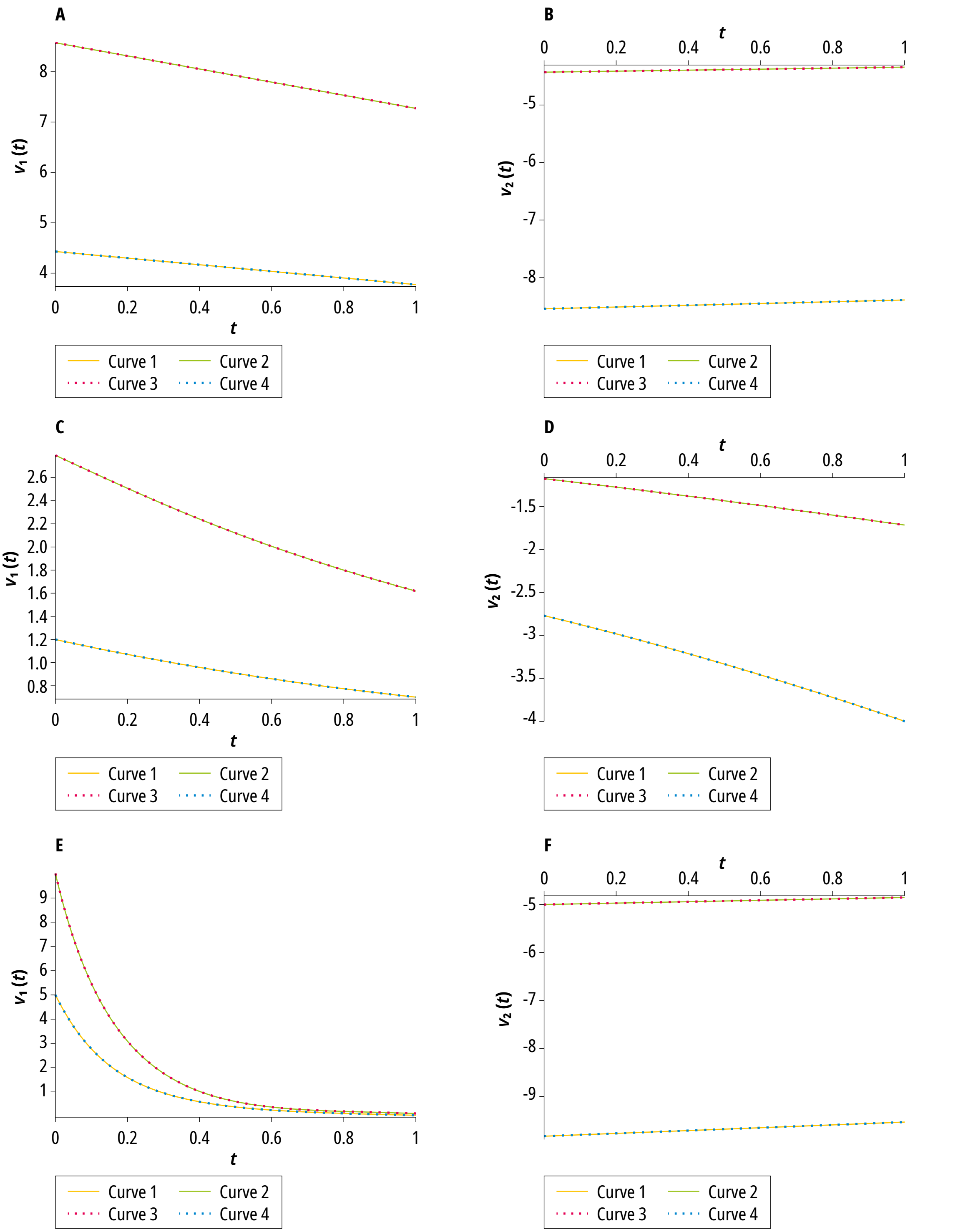

Fig. 3. Solution curves of non-linear system (13) (solid line) and linear approximation (14) (dotted line) with the initial conditions $v_0 = E^{(-1)}\,[0 \quad 5]^T$ and $v_0 = E^{(-1)}\,[2 \quad 10]^T$, when P1 (a and b), P3 (c and d), or P6 (e and f) are met.

Source: Own elaboration

Steps of Theorem 2

The steps for constructing a homeomorphism are as follows [17]-[19]:

1. Given the non-linear initial value problem $\dot{v} = r(v)$ with $v(0) = v_0$ and $M = \frac{\partial r(0)}{\partial v}$ check that $r(0) = 0$ and that M is written in the form $M = [P\ \ 0;\ 0\ \ Q]$ where the eigenvalues of P have a negative real part and the eigenvalues of Q have a positive real part.
2. Let $\phi_t(v_0)$ be the flow of the non-linear system $\dot{v} = r(v)$, and write the solution $v(t, v_0) = \phi_t(v_0) = [y(t, w_0, z_0)\ z(t, w_0, z_0)]^\top$ with $v_0 = [w_0\ z_0]^\top \in R^n$, $w_0 \in E^s$, and $z_0 \in E^u$; here, E^s and E^u are the stable and unstable subspaces of M, respectively.
3. Define $B = e^P$ and $C = e^Q$. For $v = [w\ \ z]^\top$, the 'local' transformations are defined as: $L(w, z) = [Bw\ \ Cz]^\top$ and $T(w, z) = [Bw + W(w, z)\ \ Cz + W(w, z)]^\top$ or $L(v) = e^A v$ and $T(v) = \phi_1(v)$.
4. There is an H (homeomorphism) that contains the origin such that $H \circ T = L \circ H$ (Lemma 1), where $L(w, z) = [\Phi(w, z)\ \ \Psi(w, z)]^\top$ which results from the equations
$$B\Phi(w,z) = B\Phi\big(Bw + W(w,z), Cz + Z(w,z)\big),$$
$$C\Phi(w,z) = B\Psi\big(Bw + W(w,z), Cz + Z(w,z)\big).$$

These equations can be solved by successive approximations, as follows:

$$\Phi_0(w,z) = w,$$
$$\Phi_{k+1}(w,z) = B\Phi_k(B^{-1}w + Y_1(w,z), C^{-1}z + Z_1(w,z);$$
$$\Psi_0(w,z) = z,$$
$$\Psi_{k+1}(w,z) = C^{-1}\Psi_k(Bw + W(w,z), Cz + Z(w,z).$$

The functions $Y_1(t, y, z)$ and $Z_1(t, y, z)$ are defined by the inverse of $T(y, z)$, taking as a basis

$$T^{-1}(y,z) = \Phi_{-1}(x) =$$
$$[B^{-1}y + Y_1(y,z) \quad C^{-1}z + Z_1(y,z)]^\top.$$

5. Now, let H_0 be the homeomorphism above and define

$$H = \int_0^1 L^{-s} H_0 T^s ds$$

where L^t and T^t are the one-parameter families of transformations defined by $L(x) = e^{Mt}x$ and $T^t(x) = \phi_t(x)$.

Utilizing Lemma 1, there is a neighborhood of the origin for which

$$L^t H = \int_0^1 L^{t-s} H_0 T^{s-t} ds T^t$$
$$= \int_{-t}^{1-t} L^{-s} H_0 T^s ds T^t$$
$$= \left(\int_{-t}^{0} L^{-s} H_0 T^s ds + \int_0^{1-t} L^{-s} H_0 T^s ds\right) T^t$$
$$= \int_0^1 L^{-s} H_0 T^s ds T^t$$
$$= HL^t.$$

since the Lemma 1 $H_0 = L^{(-1)} H_0$ implies that

$$\int_{-t}^{0} L^{-s} H_0 T^s ds = \int_{-t}^{0} L^{-s-1} H_0 T^{s+1} ds =$$
$$\int_{1-t}^{1} L^{-s} H_0 T^s ds.$$

Thus, $H \circ T^t = L^t\ H$ or equivalently $H \circ \phi_t(x_0) = e^{At} H(x_0)$.

Global qualitative analysis

Authors in [25] constructed a Lyapunov function for pseudo-steady state approximation (1) using a linear form, which guaranteed global asymptotic stability of the equilibrium point without colonization. Now, a new Lyapunov function characterizes the global stability of the equilibrium point with colonization.

Proposition: If $R_m > 1$, then fixed point (4) of system (1) is globally asymptotically stable in Ξ.

Proof. The Lyapunov function is built according to Theorem 3. Let $L_1: \Xi \subset R^2 \longmapsto R_+$ be a function of $C^1(\Xi)$ such that

$$L_1(x_1, x_2) = \frac{a}{2}(x_1 - \underline{x}_1)^2 + \frac{b}{2}(x_2 - \underline{x}_2)^2$$

with $[a\ b]^\top \in R^2$ and $[x_1\ \ x_2]^\top \in R^2$. Expressed in terms of the equilibrium point, the derivatives with respect to time are:

$$\star\quad \frac{dx_1}{dt} = \epsilon\left(\frac{\omega}{\epsilon}x_2 - x_1\right)$$
$$= \epsilon\left(\frac{\omega}{\epsilon}x_2 - \underline{x}_1 - (x_1 - \underline{x}_1)\right) \tag{18a}$$
$$= \omega(x_2 - \underline{x}_2) - \epsilon(x_1 - \underline{x}_1).$$

$$
\begin{aligned}
\star\quad \frac{dx_2}{dt} &= f\phi((\underline{K}-x_2)x_1-(\pi+\omega)\underline{K}x_2/(f\phi))/\underline{K}\\
&= \frac{f\phi}{\underline{K}}\left(\underline{K}(x_1-\frac{\pi+\omega}{f\phi}x_2)-x_1x_2\right)\\
&= \frac{f\phi}{\underline{K}}\left(\left(\underline{K}(x_1-\frac{\pi+\omega}{f\phi}x_2\right)-(x_1-\underline{x}_1)x_2-\underline{x}_1x_2\right)\\
&= \frac{f\phi}{\underline{K}}\left(\underline{K}\left(x_1-\left(\frac{\pi+\omega}{f\phi}+\frac{\omega}{\epsilon}\left(1-\frac{1}{R_m}\right)\right)x_2\right)-(x_1-\underline{x}_1\right.\\
&= \frac{f\phi}{\underline{K}}\left(\underline{K}\left(x_1-\frac{\omega x_2}{\epsilon}\right)-(x_1-\underline{x}_1)x_2\right)\\
&= \frac{f\phi}{\underline{K}}\left(-\frac{\underline{K}(\omega x_2-\epsilon x_1)}{\epsilon}-(x_1-\underline{x}_1)x_2\right)\\
&= -\left(\frac{f\phi}{\epsilon}\right)\frac{dx_1}{dt}-\left(\frac{f\phi}{\underline{K}}\right)(x_1-\underline{x}_1)x_2.
\end{aligned}
\tag{18b}
$$

Clearly, for all $[a \quad b]^{\top} \in R^2$ with $a > 0$ and $b > 0$, it is verified that L_1 $(\underline{x}_1, \underline{x}_2)=0$ and L_1 $(x_1, x_2) > 0$ if $(x_1, x_2) \neq (\underline{x}_1, \underline{x}_2)$. Utilizing (18a, b), the function L_1 satisfies:

$$
\begin{aligned}
\frac{dL_1}{dt} &= a(x_1-\underline{x}_1)\frac{dx_1}{dt}+b(x_2-\underline{x}_2)\left(-\frac{f\phi}{\epsilon}\frac{dx_1}{dt}-\frac{f\phi}{\underline{K}}(x_1-\underline{x}_1)x_2\right)\\
&= \left(a(x_1-\underline{x}_1)-\frac{f\phi b}{\epsilon}(x_2-\underline{x}_2)\right)\frac{dx_1}{dt}-\frac{f\phi b}{\underline{K}}(x_1-\underline{x}_1)(x_2-\underline{x}_2)x_2\\
&= \left(a(x_1-\underline{x}_1)-\frac{f\phi b}{\epsilon}(x_2-\underline{x}_2)\right)\left(\omega(x_2-\underline{x}_2)-\epsilon(x_1-\underline{x}_1)\right)\\
&\leq -\left(a\epsilon|x_1-\underline{x}_1|^2-\left(\omega|a|+f\phi|b|\left(1-\frac{x_2}{\underline{K}}\right)\right)|x_1-\underline{x}_1||x_2-\underline{x}_2|+\frac{f\phi\omega b}{\epsilon}|x_2-\underline{x}_2|^2\right)
\end{aligned}
\tag{19a}
$$

$$
\begin{aligned}
&\leq -\left(a\epsilon|x_1-\underline{x}_1|^2-(\omega|a|+f\phi|b|)|x_1-\underline{x}_1||x_2-\underline{x}_2|+\frac{f\phi\omega b}{\epsilon}|x_2-\underline{x}_2|^2\right)\\
&= -(a\epsilon|u_1|^2-(\omega a+f\phi b)|u_1||u_2|+f\phi\omega\epsilon^{-1}b|u_2|^2)=-V(u_1,u_2)
\end{aligned}
\tag{19b}
$$

with $u_i = x_i - \underline{x}_i$, $i = 1, 2$. It is assumed that the function V in (19b) is a quadratic trinomial of the form

$$
V(u_1,u_2)=c^2|u_1|^2+2cd|u_1||u_2|+d^2|u_2|^2 \equiv (c|u_1|+d|u_2|)^2. \tag{19c}
$$

Equating the coefficients having the same literal factor on both sides of (19b), we obtain

$\star\ 2cd=\omega a+f\phi b \quad \star\ d^2=f\phi\omega\epsilon^{-1}b \quad \star\ c^2=a\epsilon.$

Naturally, a and b must be positive. Intermediate equality is used to find the values of a and b as the roots of the equation:

$$2cd = \omega a + f\phi b \Leftrightarrow 4c^2d^2 = (\omega a + f\phi b)^2 \Leftrightarrow (\omega a - f\phi b)^2 = 0 \Leftrightarrow a = f\phi\omega^{-1}b.$$

Then, choose $b = \omega$ for $a = f\phi$ and the inequality (19a) becomes

$$\frac{dL_1}{dt} \leq -V(u_1, u_2) = -f\phi(\epsilon|u_1| - \omega|u_2|)^2/\epsilon \ \leq \ 0.$$

Therefore,

$$L_1(x_1, x_2) = \frac{f\phi}{2}(x_1 - \underline{x}_1)^2 + \frac{\omega}{2}(x_2 - \underline{x}_2)^2$$

is a Lyapunov function, whose graph is shown in Fig. 4 by moving the fixed point $x^{(1)}$ to the origin and taking Π (9) as its domain.

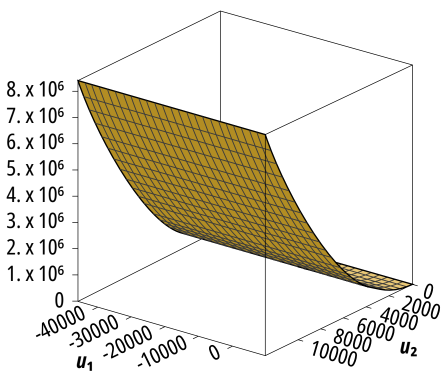

Fig. 4. Portion of the surface on the region corresponding to the Lyapunov function $L_1(u_1, u_2) = f\phi u_1^2/2 + \omega u_2^2/2$.

Source: Own elaboration

It follows from the previous calculation that $\dot{L}_1 = 0$ if and only if $x_1 = \underline{x}_1$ and $x_2 = \underline{x}_2$. By the LaSalle's invariance principle (Theorem 4), any solution of system (1) starting at (9) tends to the largest invariant subset of $S = \{x \in \Xi: \dot{L}_1(x) = 0\}$, *i.e.*, the singleton set $\{x^{(1)}\}$, since $x^{(1)}$ is locally asymptotically stable (see Proposition 3.4 in [25]). Hence, $x^{(1)} \in \Xi$ is globally asymptotically stable.

The importance of global stability for the trivial equilibrium point is because if the eradication of the mosquito population is achieved ($R_m \leq 1$), then the introduction of many mosquitoes will not lead to the recolonization. This finding has profound implications for public health; if this vector species self-extinguishes through an adequate control strategy, the epidemic outbreaks of the diseases they transmit will cease. Otherwise ($R_m > 1$), the steady-state situation is infestation by the mosquito population.

Conclusion

This paper introduced specific concepts (homeomorphism, topological equivalence, linearization, and stability) with applications to motivate mathematical population studies. In the applications of mathematical modeling, it is necessary to study not only the effect of the variations of the initial data but also the vector field by means of three approaches: analytical, qualitative, and numerical.

One of the main difficulties of the analytical approach has been the rapid growth of the mathematical complexity of both the models used to describe phenomena in sufficient detail and their analytical solutions. These difficulties arise from the presence of non-linear terms.

The qualitative approach covers local problems, particularly the flow behavior of ordinary differential equation systems near invariant sets, performing the linearization by the Taylor theorem for sufficiently differentiable scalar fields in a domain and obtaining consistent conclusions between topologically equivalent systems thanks to the Hartman-Grobman theorem. It also comprises global problems that do not require solving the systems of differential equations involved by constructing a suitable Lyapunov function.

Lastly, the numerical approach appeals to approximation and convergence using mathematical software packages (Maple, Matlab, Python, among others), capable of reproducing physical phenomena, known as numerical simulation, or giving analytical solutions to functional expressions through previous programming (*e.g.*, Picard iteration, Adomian decomposition, finite differences, among others).

The colonization process, *i.e.*, the introduction, establishment, and spread of *A. aegypti,* deserves special attention due to potential epidemic outbreaks of dengue, Zika, Chikungunya, and Mayaro. In this regard, the previous analyses play a determining role in obtaining information of biological interest about this vector.

References

[1] S. Jana and T. K. Kar, "A mathematical study of a prey–predator model in relevance to pest control," *Nonlinear Dyn.,* vol. 74, no. 3, pp. 667-683, 2013. DOI: 10.1007/s11071-013-0996-3

[2] W. M. Solano, "Dos modelos simples del método de las moscas estériles para la eliminación de la mosca de la fruta," *Pro Math.*, vol. 16, no. 31-32, pp. 165-181, 2002.

[3] J. Páez, D. Jungmann, and S. Siegmund, "Modeling and Analysis of Integrated Pest Control Strategies via Impulsive Differential Equations," *Int. J. Differ. Equ.,* vol. 2017, pp. 1-18, 2017. DOI: 10.1155/2017/1820607

[4] M. Derouich and A. Boutayeb, "An avian influenza mathematical model," *Appl. Math. Sci.*, vol. 2, no. 36, pp. 1749-1760, 2008.

[5] G. Q. Sun and Z. K. Zhang, "Global stability for a sheep brucellosis model with immigration," *Appl. Math. Comput.*, vol. 246, pp. 336-345, 2014. DOI: 10.1016/j.amc.2014.08.028

[6] J. Zhang, Z. Jin, and Y. Yuan, "Assessing the spread of foot and mouth disease in mainland China by dynamical switching model," *J. Theor. Biol.*, vol. 460, pp. 209-219, 2019. DOI: 10.1016/j.jtbi.2018.09.027

[7] A. Elaiw, "Global analysis of a cocirculating target cells HIV model with differential drug efficacy and non-linear incidence rate," *Miskolc Math. Notes,* vol. 17, no. 1, pp. 231-244, 2016. DOI: 10.18514/MMN.2016.1287

[8] J. M. Epstein, *Nonlinear Dynamics, Mathematical Biology, and Social Science: Wise Use of Alternative Therapies*. United States: CRC Press, 2018.

[9] J. M. Krueger, "Parameter Estimation Methods for Ordinary Differential Equation Models with Applications to Microbiology," Ph.D. dissertation, Virginia Tech, Blacksburg (VA), 2016.

[10] T. T. Yusuf and F. Benyah,"Optimal control of vaccination and treatment for an SIR epidemiological model," *World J. Model. Simul.*, vol. 8, no. 3, pp. 194-204, 2012.

[11] T. Asada, "Modelando la inestabilidad financiera," *Rev. Eur. Econ. Polit. Econ.* , vol. 9, pp. 215-232, 2012.

[12] S. Ghosh, A. Deb, and G. Sarkar, "A new recursive method for solving state equations using Taylor series," *Int. J. Electr. Comput. Eng.*, vol. 1, no. 2, pp. 22-27, 2012.

[13] A. Abad, R. Barrio, M. Marco-Buzunariz, and M. Rodríguez, "Automatic implementation of the numerical Taylor series method: A Mathematica and Sage approach," *Appl. Math. Comput.*, vol. 268, pp. 227-245, 2015. DOI: 10.1016/j.amc.2015.06.042

[14] R. Barrio, M. Rodríguez, A. Abad, and F. Blesa, "Breaking the limits: the Taylor series method," *Appl. Math. Comput.*, vol. 217, no. 20, pp. 7940-7954, 2011. DOI: 10.1016/j.amc.2011.02.080

[15] S. N. Li and B. Y. Cao, "On defects of Taylor series approximation in heat conduction models," *Int. J. Heat Mass Transf.*, vol. 98, no. 2016, pp. 824-832, 2016. DOI: 10.1016/j.ijheatmasstransfer.2016.03.067

[16] V. Šátek, F. Kocina, J. Kunovsky, and A. Schirrer, "Taylor series based solution of linear ode systems and matlab solvers comparison," *IFAC*, vol. 48, no. 1, pp. 693-694, 2015. DOI: 10.1016/j.ifacol.2015.05.210

[17] L. Perko, *Differential Equations and Dynamical Systems*. New York: Springer-Verlag, 1996.

[18] D. S. Graça, N. Zhong, and H. S. Dumas, "The connection between computability of a non-linear problem and its linearization: The Hartman Grobman theorem revisited," *Theor. Comput. Sci.*, vol. 457, no. 2012, pp. 101-110, 2012. DOI: 10.1016/j.tcs.2012.07.013

[19] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. New York: Academic Press, 2012.

[20] S. M. O'Regan, T. C. Kelly, A. Korobeinikov, M. J. O'Callaghan, and A. V. Pokrovskii, "Lyapunov functions for SIR and SIRS epidemic models," *Appl. Math. Lett.*, vol. 23, no. 4, pp. 446-448, 2010. DOI: 10.1016/j.aml.2009.11.014

[21] G. H. Moore, "The evolution of the concept of homeomorphism," *Hist. Math.*, vol. 34, no. 3, pp. 333-343, 2007. DOI: 10.1016/j.hm.2006.07.006

[22] J. E. Marsden, A. J. Tromba, and M. L. Mateos, *Cálculo vectorial* (vol. 69). Addison-Wesley Iberoamericana, 1991.

[23] S. Wiggins, *Introduction to applied non-linear dynamical systems and chaos* (Vol. 2). Springer Science & Business Media, 2003.

[24] H. K. Khalil and J. W. Grizzle. *Non-linear systems* (vol. 3). Upper Saddle River, NJ: Prentice Hall, 2002.

[25] J. A. Olarte-García and A. Muñoz-Loaiza, "Un modelo de crecimiento poblacional de *Aedes aegypti* con capacidad de carga logística," *Rev. Mat.*, vol. 25, no. 1, pp. 79-113, 2018. DOI: 10.15517/RMTA.V1I25.32233

[26] H. M. Yang, M. D. L. D. G. Macoris, K. C. Galvani, M. T. M. Andrighetti, and D. M. V. Wanderley, "Assessing the effects of temperature on the population of *Aedes aegypti*, the vector of dengue," *Epidemiol. Infect.*, vol. 137, no. 8, pp. 1188-1202, 2009. DOI: 10.1017/S0950268809002040